STURM-LIOUVILLE PROBLEMS

We want to study the bd value problem on interval

$$\begin{cases} -(pu')' + cu' + Vu = f \\ \alpha_1 u(0) + \beta_1 u'(0) = a \\ \alpha_2 u(1) + \beta_2 u'(1) = b \end{cases}$$

with

∘) $p, c, V \in C^0([0,1])$, $f \in L^2([0,1])$
real valued

∘) $p(x) \geq \alpha > 0 \quad \forall x \in [0,1]$

$\alpha_1^2 + \beta_1^2, \; \alpha_2^2 + \beta_2^2 \neq 0$ (non-deg. conditions)

EXAMPLES

1. Dirichlet BVP

$$\begin{cases} -(pu')' + Vu = f \\ u(0) = u(1) = 0 \end{cases} \qquad \left(\begin{cases} -u'' + Vu = f \\ u(0) = u(1) = 0 \end{cases} \right.$$

2. Neumann BVP

$$\begin{cases} -(pu')' + Vu = f \\ u'(0) = u'(1) = 0 \end{cases}$$

We start with Dirichlet

Q : Solution in which sense?

CLASSICAL SOLUTION: $u \in C^2([0,1])$ solving the problem in the "classical sense"

Assume p, u, V, f are C^∞, take $\varphi \in C_0^\infty([0,1])$
multiply the eq by φ and $\int$

$$\int -(pu')'\varphi + \int Vu\varphi = \int f\varphi$$

$\Downarrow$ int. by parts: bd term: $(pu')\varphi\Big|_0^1 = p(1)u'(1)\varphi(1) - p(0)u'(0)\varphi(0)$
No information on $u'(0)$ & $u'(1)$:
put $\varphi(1) = \varphi(0) = 0$

$$\boxed{\int (pu')\varphi' + \int Vu\varphi = \int f\varphi \quad \forall \varphi}$$

WEAK PROBLEM

If $p \in C^1$, $V, f \in C^0$, $u \in C^2$ solves the weak problem and $u(0) = u(1) = 0$, then

$$\int \left(-(pu')' + Vu - f\right)\varphi = 0 \quad \forall \varphi \in C_0^\infty$$

$$\Downarrow$$

$$-(pu')' + Vu - f = 0 \quad \text{in } [0,1]$$

get again a classical solution

Advantage of WP: it makes sense in lower regularity than classical problem:

It makes sense $\forall\ u, u' \in L^2$, $\varphi, \varphi' \in L^2$ + b.c.
$p, V \in L^\infty$

to incorporate

GENERAL STRATEGY:

(1) Find the right functional space to study the problem (incorporate bc + minimal regularity required)

(2) solve WP

(3) regularity of weak solution

(4) back to classical solution

1. How to find the space

$$H^1([a,b]) = \left\{ u \in L^2 : \exists\ g \in L^2 \text{ st } \int u\varphi' = -\int g\varphi \quad \forall \varphi \in C_0^\infty \right\}$$

If $u \in H^1$, we say that $g \equiv u'$ is the WEAK DERIVATIVE fu

H^1 is Hilbert space with scalar product

$$\langle u, f \rangle_{H^1} := \langle u, f \rangle_{L^2} + \langle u', f' \rangle_{L^2}$$

$$\langle u, f \rangle_{L^2} = \int u(x) f(x)\, dx$$ (Today: real Hilbert spaces)

In corporate b.c. in functional space

$$H_0^1([0,1]) = \{ u \in H^1 : u(1) = u(0) = 0 \}$$

makes sense $H^1 \hookrightarrow C^{1/2}([0,1]) \overset{\text{compact}}{\hookrightarrow} L^2$

<u>FACT</u>: ∘) $u \in H^1 \Rightarrow \exists \tilde{u} \in C([0,1])$ st $u = \tilde{u}$ a.e. and

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)\,dt \qquad \forall x,y \in [0,1]$$

$\tilde{u}$: continuous representative

∘) $H_0^1 \subseteq H^1$ closed subspace ⇝ Hilbert

∘) $H_0^1 = \overline{C_0^\infty}^{H^1}$

∘) POINCARÉ INEQUALITY: $\int_{[0,1]} |u(x)|^2 \leq c\, \|u'\|_{L^2} \quad \forall u \in H_0^1([0,1])$

<u>proof</u> $|u(x)| = |u(x) - u(0)| = \left|\int_0^x u'(t)\,dt\right| \leq \|u'\|_{L^2}$

CONSEQUENCE: on H_0^1, the norm $\|u\|_{L^2} + \|u'\|_{L^2} \sim \|u'\|_{L^2}$

If we find a solution of WP in H_0^1 ⇝ we have a function incorporating BC.

Def $u \in H_0^1([0,1])$ is a WEAK SOLUTION if

$$\int p\, u' \varphi' + \int r\, u\, \varphi = \int f \varphi \qquad \forall \varphi \in H_0^1([0,1])$$

•) How to find weak solution

Functional analysis argument: define

$$a(u,\varphi) = \int p u' \varphi' + \int V u \varphi$$

$$F(\varphi) = \int f \varphi$$

Then WP $\Leftrightarrow$ given $f \in L^2$, $\exists$ $u \in H^1_0$:

$$\underbrace{a(u,\varphi)}_{\text{bilinear continuous form on } H^1_0} = F(\varphi) \quad \forall \varphi \in H^1_0$$

Idea $\quad a(u,\varphi) = \langle Au, \varphi \rangle$

$F(\varphi) = \langle g, \varphi \rangle_{H^1_0}$

so if A invertible, great! $\rightsquigarrow$ coercivity of a

Thm (Lax-Milgram) H real Hilb. space

$$a : H \times H \longrightarrow \mathbb{R} \quad \text{bilinear}$$

•) continuous: $|a(u,v)| \le C_1 \|u\| \|v\| \quad \forall u,v \in H$

•) coercive: $a(u,u) \ge C_2 \|u\|^2 \quad \forall u \in H$

Then $\forall F \in H'$, $\exists!$ $u \in H$: $a(u,v) = F(v)$ $\forall v \in H$

proof By Riesz: $F(v) = \langle f, v \rangle_H$ for some $f \in H$

a continuous $\Rightarrow$ $a(u,v) = \langle Au, v \rangle_H$

with $A \in \mathcal{L}(H)$

coercivity $\Rightarrow$ $\langle Au, u\rangle \geq c_2 \|u\|^2$

CLAIM: A is invertible.

Indeed $\begin{cases} \ker A = 0 \\ \operatorname{Im} A \text{ closed} \\ \ker A^* = 0 \end{cases}$ $\quad (c_2\|u\|^2 \leq \|Au\|\,\|u\|)$ $\quad (c_2\|u\|^2 \leq \langle u, A^* u\rangle)$

$$X = (\ker A^*)^\perp = \overline{\operatorname{Im} A} = \operatorname{Im} A$$

$\rightsquigarrow$ A cont. & bijective $\Rightarrow$ $\exists A^{-1} \in \mathcal{L}(H)$

$\rightsquigarrow$ $\langle Au, v\rangle = \langle f, v\rangle \quad \forall v \in H$

$\rightsquigarrow$ $Au = f \Rightarrow u = A^{-1} f$

unique sol of problem: $\|u\|_H \leq C_1 \|f\|_H$ $\square$

EXERCISE: extend statement and proof to complex Hilbert spaces
$a: H \times H \to \mathbb{C}$ sesquilinear.

Application to Sturm-Liouville

$$a: H^1_0 \times H^1_0 \to \mathbb{R}, \quad a(u,v) = \int p u' v' + \int V u v$$

$$F: H^1_0 \to \mathbb{R}, \quad F(v) = \int f \cdot v, \quad f \in L^2$$

Then

(1) a bilinear ✓

(2) a continuous: $|a(u,v)| \leq \|p\|_\infty \|u'\|_{L^2} \|v'\|_{L^2} + \|V\|_\infty \|u\|_{L^2} \|v\|_{L^2}$ ✓

(3) a coercive:

$$a(u,u) = \int p(u')^2 + \int V u^2 \overset{?}{\geq} C\left(\|u'\|_{L^2}^2 + \|u\|_{L^2}^2\right)$$

Recall: $p(x) \geq \alpha > 0$, but no sign on V.
If V is too negative, not granted!

$\rightsquigarrow$ for the moment add additional assumption:

$$V(x) \geq c > 0$$

then a coercive!

(4) $F \in (H_0^1)'$: $|F(v)| \leq \|f\|_{L^2} \|v\|_{L^2}$
$\leq \|f\|_{L^2} \|v\|_{H_0^1}$

Lax-Milgram: $\exists!\ u \in H_0^1$:

$$a(u,v) = F(v) \qquad \forall v \in H_0^1$$

$\rightsquigarrow$ we have solved WP!

⚠ When applying Riesz, we write $F(v) = \int f v$ as

$$F(v) = \langle g, v\rangle_{H_0^1} = \int g v + g' v'$$

in general $g \neq f$

Who is g? $\rightsquigarrow \int -g''v + gv = \int fv \quad \forall v$

$\rightsquigarrow \begin{cases} -g'' + g = f \\ g(0) = g(1) = 0 \end{cases}$ g is w.s. of this Dirichlet b.c.

<u>Recap:</u> so far $\forall f \in L^2$, $\exists!\, u$ ws of

$$\int p u' v' + \int \sigma u v = \int f v \qquad \forall v \in H^1_0$$

assuming $\sigma(x) \geq c > 0$

We can define

$$T : L^2 \xrightarrow{\tilde{T}} H^1_0 \xhookrightarrow{i} L^2$$

$$f \longrightarrow \tilde{T}f = u \text{ sol of WP} \hookrightarrow Tf$$

Prop T linear, continuous, compact, $T = T^*$

<u>proof</u> lin ✓

<u>Continuity</u>: $\|\tilde{T}f\|^2_{H^1_0} = \langle \tilde{T}f, \tilde{T}f \rangle_{H^1_0}$

$$= \int (\tilde{T}f)'(\tilde{T}f)' + \int (\tilde{T}f)(\tilde{T}f)$$

$p(x) \geq \alpha > 0$
$\sigma(x) \geq c > 0$

$$\leq \frac{1}{\alpha} \int p(x) (\tilde{T}f)'(\tilde{T}f)' + \frac{1}{c} \int \sigma(x) (\tilde{T}f)(\tilde{T}f)$$

$$\leq \frac{1}{\min(\alpha, c)} \int p (\tilde{T}f)'(\tilde{T}f)' + \int \sigma (\tilde{T}f)(\tilde{T}f)$$

$\tilde{T}f$ sol of WP: $\int p (\tilde{T}f)' v' + \int V (\tilde{T}f) v = \int f v \quad \forall v \in H^1_0$

Now put $v = \tilde{T}f$

$$= \frac{1}{\min(\alpha, c)} \int (\tilde{T}f) \cdot f$$

$$\leq C \langle \tilde{T}f, f \rangle_{L^2} \leq C \|\tilde{T}f\|_{L^2} \|f\|_{L^2}$$

$$\leq C \|\tilde{T}f\|_{H^1_0} \|f\|_{L^2}$$

$$\Rightarrow \begin{cases} \langle \tilde{T}f, f \rangle \geq 0 & \forall f \in L^2 \\ \|\tilde{T}f\|_{H^1_0} \leq C \|f\|_{L^2} \end{cases}$$

$\rightsquigarrow \quad \tilde{T} : L^2 \to H^1_0$ continuous

$\rightsquigarrow \quad T = i \circ \tilde{T}$ continuous $\qquad i : H^1_0 \hookrightarrow L^2$

<u>Compact</u>: $T : L^2 \to H^1_0 \overset{i}{\hookrightarrow} L^2$ is compact

compact

<u>$T = T^*$</u> in L^2 $\qquad Tf$ solves: $\int p (Tf)' v' + \int V (Tf) v = \int f v \quad \forall v$

if $v = Tg$:

$$\langle f, Tg \rangle = \int p (Tf)' (Tg)' + \int V (Tf)(Tg)$$

symmetric in f & g

$\langle g, Tf \rangle$ //

" $\langle Tf, g \rangle$ $\qquad$ by reality of $\langle , \rangle$

$\blacksquare$

Now we want to drop the assumption $V(x) \geq c > 0$

Just take $V \in C^0([0,1])$ and solve

$$(D) \begin{cases} -(pu')' + Vu = f \\ u(0) = u(1) = 0 \end{cases}$$

Building on previous case: put $M := 2\|V\|_\infty$ and consider

$$(D_M) \quad \begin{cases} -(pu')' + (V+M)u = f \\ u(0) = u(1) = 0 \end{cases}$$

Now: $V(x) + M =: V_M(x) \geq c > 0$

Denote by T_M the sol map of (D_M)
$T_M : L^2([0,1]) \to L^2([0,1])$ compact & selfadj.

so $T_M f$ is the sol of (D_M) with given data f

Go back to (D): write it as

$$(\tilde{D}_M) \begin{cases} -(pu')' + Vu + Mu = f + Mu \\ u(0) = u(1) = 0 \end{cases}$$

<u>key remark</u>:

u sol of (D) with datum f $\iff$ u sol of $(\tilde{D}_M)$ with datum $f + Mu$

$$u = Tf = T_M(f + Mu)$$

$$\rightsquigarrow \quad T_M f = u - M T_M u = (\mathbb{1} - M T_M) u$$

$\rightsquigarrow$ if $(\mathbb{1} - M T_M)$ invertible

$$u = (\mathbb{1} - M T_M)^{-1} \circ T_M f$$

T_M compact $\rightsquigarrow$ apply Fredholm theory:

$$\operatorname{Im}(\mathbb{1} - M T_M) = L^2 \iff \ker(\mathbb{1} - M T_M) = \{0\}$$

Moreover $\operatorname{Im}(\mathbb{1} - M T_M) = \ker(\mathbb{1} - M T_M)^{\perp}$ $(T_M = T_M^*)$

So let us compute $\ker(\mathbb{1} - M T_M)$

$$u \in \ker(\mathbb{1} - M T_M) \iff u = M T_M u \iff \frac{1}{M} u = T_M u$$

$\iff$ $\frac{1}{M} u$ solves (D_M) with given datum u

$$\begin{cases} -\frac{1}{M}(pu')' + (V + M)\frac{1}{M} u = u \\ u(0) = u(1) = 0 \end{cases}$$

$$\iff \begin{cases} -(pu')' + Vu = 0 \\ u(0) = u(1) = 0 \end{cases} \qquad (*)$$

homogeneous problem: it might or might not have a solution

By Fredholm theory:

<u>1st result</u>

$\forall f \in L^2$, $\exists!$ sol of (D) $\iff$ $\nexists$ sol $\neq 0$ of $(*)$

<u>2nd result</u>

(D) has solution $\iff$ $\langle f, v \rangle = 0$ $\forall$ v sol of $(*)$

<u>Indeed</u>: from $(\mathbb{I} - M T_M)\, u = T_M f$

(D) has solution $\iff T_M f \in \operatorname{Im}(\mathbb{I} - M T_M)$

$\iff T_M f \perp \ker(\mathbb{I} - M T_M)$

i.e. $\forall v \in \ker(\mathbb{I} - M T_M)$, we have $v = M T_M v$

$$0 = \langle T_M f, v \rangle = \langle f, T_M v \rangle = \frac{1}{M} \langle f, v \rangle$$

$\iff f \perp \ker(\mathbb{I} - M T_M)$

$\iff f \perp v$ sol of $(*)$

<u>EXAMPLE 1</u> $p(x) \equiv 1$, $V(x) \equiv -\alpha^2$ constant

$$(D) \begin{cases} -u'' - \alpha^2 u = f \\ u(0) = u(1) = 0 \end{cases}$$

Check the homogeneous problem

$$(*) \quad \begin{cases} u'' + \alpha^2 u = 0 \\ u(0) = u(1) = 0 \end{cases}$$

$\alpha \neq n\pi$, $n \in \mathbb{Z}$ $\rightsquigarrow \nexists$ sol

$\alpha = n\pi \Rightarrow B \sin(n\pi x)$ is the unique sol of $(*)$

harmonic oscillator

$u(x) = A \cos(\alpha x) + B \sin(\alpha x)$: impose BC.

Conclusion : ◦) $\forall f \in L^2$ $\exists!$ sol of (D) $\Leftrightarrow$ $\alpha \neq n\pi$, $n \in \mathbb{Z}$

◦) if $\alpha = n\pi$, $n \in \mathbb{Z}$, then $\exists$ sol of (D) $\Leftrightarrow$ $\langle f, \sin(n\pi x) \rangle = 0$

EXAMPLE 2 Consider $V(x) \geq 0$

$$\begin{cases} -(pu')' + Vu = f \\ u(0) = u(1) = 0 \end{cases}$$

again consider hom. problem: $\begin{cases} -(pu')' + Vu = 0 \\ u(0) = u(1) = 0 \end{cases}$ $(**)$

Let u a ws of $(**)$, then

$$\int p(u')^2 + \int V u^2 = 0 \Rightarrow \int p(u')^2 = 0$$

$\Rightarrow u = \text{const} \xRightarrow{u \text{ fulfills BC}} u = 0$

So if $V(x) \geq 0$ then $T = (\mathbb{1} - MT_M)^{-1} T_M \in \mathcal{L}(L^2)$

$T = T^*$ and T compact

Conclusion $\forall f \in L^2$, $\exists!$ u weak sol of (D)

•) Return to classical solutions

Actually all weak sol are classical solutions if $f, V \in C^0$, $p \in C^1$. Indeed consider

$$\begin{cases} -(pu')' + Vu = f \\ u(0) = u(1) = 0 \end{cases}$$

$$u \text{ ws} \iff \int p u' v' + \int V u \cdot v = \int f v \quad \forall v \in H^1_0$$

$$\rightsquigarrow \int p u' v' = -\int (Vu - f)\, v \quad \forall v \ C_0^\infty$$

$\rightsquigarrow$ the function $g := Vu - f \in L^2$ is the weak derivative of pu' : $\int w v' = -\int g v \quad \forall v$

$$\rightsquigarrow p u' \in H^1 \hookrightarrow C^0 \quad \rightsquigarrow \quad u' = \underbrace{\frac{1}{p}}_{C^1} \cdot \underbrace{p u'}_{H^1} \in H^1$$

$$\rightsquigarrow u \in H^2$$

Moreover if $f, V \in C^0$,

$$\rightsquigarrow (pu')' = g = Vu - f \in C^0 \quad (\text{if } V, f \in C^0)$$

$$\rightsquigarrow pu' \in C^1 \rightsquigarrow u' = \underbrace{\frac{1}{p}}_{\in C^1} \cdot \underbrace{pu'}_{\in C^1} \rightsquigarrow u' \in C^1 \rightsquigarrow u \in C^2$$

<u>Conclusion</u> : $\forall f \in L^2$, $\exists!\, u \in H^2$ solving wp

If $V, f \in C^0$, $p \in C^1$ $\Rightarrow$ $u \in C^2$ is classical solution

Neumann problem

As before, $V \in C^0$, $p \in C^1$, $p(x) \geq \alpha > 0$

$$(N) \quad \begin{cases} -(pu')' + Vu = f \\ u'(0) = u'(1) = 0 \end{cases}, \quad f \in L^2$$

Weak formulation: if u is a classical sol ($u \in C^2$) multiply by $v \in C^\infty$ (not necessarily with comp supp) and $\int$

$$-\int (pu')' v + \int Vuv = \int fv$$

$$\Rightarrow \int pu'v' - \underbrace{(pu'v)\Big|_0^1}_{p(1)u'(1)v(1) - p(0)u'(0)v(0) = 0} + \int Vuf = \int fv \quad \forall v \in C^0$$

Now have information on $u'(1) = u'(0) = 0 \rightsquigarrow$ no constraint on $v(0)$ & $v(1)$

It makes sense to work in H^1:

Def $u \in H^1$ is a WS for (N) iff

$$\underbrace{\int pu'v' + \int Vuv}_{a(u,v)} = \underbrace{\int fv}_{F(v)} \quad \forall v \in H^1$$

Apply Lax-Milgram: again if $V(x) \geq c > 0$ we have $a : H^1 \times H^1 \to \mathbb{R}$ is continuous and coercive

As in the previous case $F : H^1 \to \mathbb{R}$ continuous

$\rightsquigarrow \forall f \in L^2$, $\exists!\, u \in H^1$ sol of WP, provided $V(x) \geq c > 0$

Back to classical solution:

$$\int p u' v' = -\int (V u - f) v \qquad \forall v \in H^1$$

$\rightsquigarrow$ pu' has weak derivative $g := Vu - f \in L^2$

$\rightsquigarrow$ $pu' \in H^1$ $\rightsquigarrow$ $u' = \underbrace{\frac{1}{p}}_{\in C^1} \cdot \underbrace{pu'}_{\in H^1} \in H^1$

$\rightsquigarrow$ $u' \in C^0$ (so it makes sense $u'(0)$ and $u'(1)$)

$\rightsquigarrow$ $u \in H^2$

$$(WP) \Rightarrow \int \left(-(pu')' + Vu - f\right) v - \underbrace{pu'v\Big|_0^1}_{p(1)u'(1)v(1) - p(0)u'(0)v(0)} = 0 \quad \forall v \in H^1$$

take $v \in C_c^\infty$ $\Rightarrow$ $\int \left(-(pu')' + Vu - f\right) v = 0 \quad \forall v \in C_0^\infty$

$\rightsquigarrow$ $-(pu')' + Vu - f = 0$ a.e. $(\ast)$

$\rightsquigarrow$ $p(1)u'(1)v(1) = p(0)u'(0)v(0) \qquad \forall v \in H^1$

As $v(0)$ & $v(1)$ arbitrary $\Rightarrow$ $u'(1) = 0 = u'(0)$

Back to $(\ast)$: If $f \in C^0$: $-(pu')' = \underbrace{f - Vu}_{\in C^0}$

$\rightsquigarrow$ $-pu' \in C^1$

$\rightsquigarrow$ $u' = \frac{1}{p} pu' \in C^1$ $\rightsquigarrow$ $u \in C^2$

Conclusion: $\forall f \in L^2$, $\exists! \, u \in H^2$ solving WP

If $f \in C^0$ $\Rightarrow$ $u \in C^2$ is classical solution

What about general $V(x)$ (not $V(x) \geq c > 0$)?
Proceed as in the previous case:

$$(N_M) \quad \begin{cases} -(pu')' + (V+M)u = f \\ u'(0) = u'(1) = 0 \end{cases}, \quad \text{with } \underbrace{V+M \geq c > 0}_{\text{coercivity}}$$

$\rightsquigarrow$ $\forall f \in L^2$, $\exists!$ u_M sol of (N_M) : $u_M = T_M(f)$

u solves (N) $\iff$ $u = T_M(f + Mu)$

$\iff$ $(\mathbb{1} - MT_M)u = T_M f$

(as in the previous case)
$\iff$ $\langle f, v\rangle = 0$ $\forall v$ sol of homogeneous problem

$$\begin{cases} -(pv')' + Vv = 0 \\ v'(0) = 0 = v'(1) \end{cases}$$

EXERCISE: fills the details!

What about homogeneous problem?

Easy case: $V = 0$

$$(N2) \quad \begin{cases} -(pu')' = f \\ u'(0) = 0 = u'(1) \end{cases} \quad \text{has sol} \iff \langle f, v\rangle = 0 \ \forall v \text{ solving } \begin{cases} -(pv')' = 0 \\ v'(0) = v'(1) = 0 \end{cases}$$

So, if v is ws of hom prob: $\int p(v')^2 = 0 \Rightarrow$
$\rightsquigarrow$ $v' = 0$ a.e. $\rightsquigarrow$ $v = \text{const}$
We need: $\int f \cdot c = 0$ $\forall c$ constant $\rightsquigarrow$ $\int f = 0$

Spectral analysis

Study the problem

$$(D_\lambda) \quad \begin{cases} -(pu')' + Vu = \lambda u \\ u(0) = u(1) = 0 \end{cases}$$

Assume $V(x) \geq 0$ (otherwise replace $V \rightsquigarrow V+M$, $\lambda \rightsquigarrow \lambda+M$)

u sol of (D_λ) $\iff$ $T(\lambda u) = u$

$\iff$ $Tu = \frac{1}{\lambda} u$

$\iff$ u eigenfunction of T with eigenvalue $\frac{1}{\lambda}$

From spectral thm for compact selfadjoint ops

$\exists \{e_k\}_k$ ON basis of $L^2([0,1])$ so that

$Te_k = \mu_k e_k$, μ_k are real

Moreover:

$p(x) \geq \alpha > 0$
$V(x) \geq 0$

$$\langle Tu, u\rangle = \int Tu \cdot u = \int p (Tu)'^2 + \int V (Tu)^2 \geq 0$$

Tu weak sol with given cond. u

$\int p (Tu)' v' + \int V(Tu) v = \int u v \quad \forall v \in H^1_0$

put $v = Tu$

$\rightsquigarrow$ $\langle Tu, u\rangle \geq 0$ $\forall u \in L^2$ $\rightsquigarrow$ eigenvalues are ≥ 0

Moreover $\ker T = \{0\}$: indeed by Fredholm

(D) has 1! sol $\Leftrightarrow$ the homog. problem $\begin{cases} -(pu')' + Vu = 0 \\ u(0) = u(1) = 0 \end{cases}$
has only the trivial sol.

But if $V(x) \geq 0$ this is true: so $Tf = 0 \Rightarrow f = 0$

$\rightsquigarrow$ $\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots > 0$

$\rightsquigarrow$ $\{e_n\}_n$ ON basis of eigenvectors with

$$T\vec{e_n} = \mu_n \vec{e_n}$$

So we have proved

Thm take $p \in C^1$, $p \geq \alpha > 0$, $V \in C^0$, $V \geq 0$.
then $\exists$ seq $(\lambda_n)_{n\geq 1}$ of real numbers and
Hilbert base $(e_n)_{n\geq 1}$ of $L^2([0,1])$ st

$\vec{e_n} \in C^2$ $\forall n$ and

$$\begin{cases} -(p e_n')' + V e_n = \lambda_n e_n \\ e_n(0) = e_n(1) = 0 \end{cases}$$

and $\lambda_n \to +\infty$ for $n \to \infty$

proof Just put $\lambda_n = \frac{1}{\mu_n} \to +\infty$ and note

that $\vec{e}_n \in H^1_0 \hookrightarrow C^0$

and $-(p\, e_n')' = \lambda e_n - V e_n \in C^0$

$\rightsquigarrow e_n' \in C^2$

$\square$

EXAMPLE: $\begin{cases} -u'' = \lambda u \\ u(0) = u(1) = 0 \end{cases}$

we know there are non trivial solutions only for $\lambda_k = k^2\pi^2$, and $u_k(x) = \sin(k\pi x)$ is only solution

$\rightsquigarrow \{ e_k(x) := \sqrt{2}\, \sin(k\pi x) \}$ orthonormal basis

Problems on the line

So far we used in a crucial way that the problem is on $[0,1]$ $\rightsquigarrow$ this gives T is compact, as $H^1_0 \hookrightarrow L^2$ is compact.

If we extend $[0,1]$ to $\mathbb{R}$ the compactness is lost.

$\langle u,v \rangle = \int_{\mathbb{R}} u'v' + \int_{\mathbb{R}} uv$

$H^1(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) ;\ \exists\, u' \text{ weak derivative in } L^2 \}$

FACT $\circ$) $u \in H^1(\mathbb{R}) = \{ u \in L^2 : \ \hat{u}(\xi)\,(1+|\xi|) \in L^2 \}$

Indeed: $u \in L^2 \iff \hat{u}(\xi) = \int e^{ix\xi}\, u(x)\,dx \in L^2$

$u' \in L^2 \iff \xi\, \hat{u}(\xi) \in L^2$

$\circ$) $u \in H^1(\mathbb{R}) \implies u(x) \to 0$ if $|x| \to +\infty$

Indeed: $u(x) = \int e^{-ix\xi}\, \hat{u}(\xi)\, d\xi \longrightarrow 0 \quad x \to +\infty$ Riemann Lebesgue

$\hat{u} \in L^1$: $\int |\hat{u}(\xi)| \le \left(\int \frac{1}{1+\xi^2} \right)^{1/2} \int (1+\xi^2)\, |\hat{u}(\xi)|^2 < +\infty$

•) $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is not compact

Indeed take $\varphi \in C_0^\infty(\mathbb{R})$, put

$$\varphi_n(x) = \varphi(x-n)$$

$\Rightarrow \varphi_n \in H^1(\mathbb{R})$, $\quad \|\varphi_n\|_{H^1}^2 = \int_{\mathbb{R}} |\varphi_n'(x)|^2 + \int_{\mathbb{R}} |\varphi_n(x)|^2 = \|\varphi\|_{H^1}$

If $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ was compact, we would have

$$\varphi_{n_k} \longrightarrow \varphi_\infty \quad \text{in } L^2$$

But $\varphi_{n_k}(x) \longrightarrow 0$ pointwise, whereas $\|\varphi_{n_k}\|_{L^2} = c$

So consider the problem on the line $\nsubseteq$

$$\begin{cases} -u'' + u = f \\ u(x) \longrightarrow 0 \qquad |x| \to \infty \end{cases}$$

weak formulation: u classical sol, multiply by $v \in C_0^\infty(\mathbb{R})$ getting

$$\int_{\mathbb{R}} u'v' + \int_{\mathbb{R}} uv = \int_{\mathbb{R}} fv \qquad \forall\, v \in C_0^\infty(\mathbb{R})$$

It makes sense $\forall\, u, v \in H^1(\mathbb{R})$

(WP) $\quad u \in H^1$ sol of $\quad \underbrace{\int u'v' + \int uv}_{\langle u,v \rangle} = \underbrace{\int fv}_{F(v):\, H^1(\mathbb{R}) \to \mathbb{R} \text{ continuous}} \quad \forall v \in H^1$

(1) $\exists$ weak solution: by Riesz theorem

$$\forall\, f \in L^2,\ \exists!\, u \in H^1 : \quad \langle u, v \rangle_{H^1} = F(v) \quad \forall\, v \in H^1$$

(2) $T : f \longmapsto Tf = u$ solution of (WP)

T linear, bd, not compact, $T = T^*$

(3) What about $\sigma(T)$? Recall that Fourier transform

is isometry of L^2: $\|\hat{u}\|_{L^2} = \|u\|_{L^2}$

$\to \mathcal{F}(-u'') = \xi^2 \, \hat{u}(\xi)$

take $-u'' + u = f$ and Fourier transform it

$$\xi^2 \hat{u} + \hat{u} = \hat{f} \quad \rightsquigarrow \quad \hat{u}(\xi) = \frac{\hat{f}(\xi)}{1+\xi^2}$$

$$\rightsquigarrow \quad u(x) = \int e^{ix\xi} \, \frac{\hat{f}(\xi)}{\xi^2+1} \, d\xi$$

Then $T = \mathcal{F}^{-1} \hat{T} \mathcal{F}$, $\hat{T}\hat{f} = \frac{1}{\xi^2+1} \hat{f}(\xi)$ on L^2

(multiplication operator)

$$\sigma(M) = \overline{\operatorname{Im}\left(\frac{1}{\xi^2+1}\right)} = [0,1] = \sigma_c(T)$$

$\square$